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The two states being 1 and 2 (indicated by superscripts) and variables x_i (i = 1,2, ..., k) the problem is to find the coefficients f_i so that the identity obtains

$$(1) \quad \prod_{i=1}^{k} x_i^2 - \prod_{i=1}^{k} x_i^1 \equiv \sum_{i=1}^{k} f_i \, (x_i^2 - x_i^1),$$

It is obvious that (i) f_i must be a homogenious sum product of all the variables x_j^2 and x_j^1 of dimension (k - 1), (ii) that the sum of the coefficients in each f_i must be 1, (iii) the solution is not unique, without the imposition of conditions. As to what these conditions should be, consider the case of two variables

$$(2) \quad x_1^2 \, x_2^2 - x_1^1 \, x_2^1 \equiv f_1(x_1^2 - x_1^1) + f_2(x_2^2 - x_2^1)$$

It turns out that one fascinating set of values of f_1 and f_2 is

$$(3) \quad f_1 = (x_2^1 + x_2^2)/2, \; f_2 = (x_1^1 + x_1^2)/2.$$

It will be seen that with these values in (2) the terms on the right, namely $x_1^2 \, x_2^1$ and $x_1^1 \, x_2^2$, mixed in states, cancel out. But this property also obtains in, say

$$(4) \quad f_i = x_2^2, \; f_2 = x_1^1,$$

Showing that the solution is not unique. But (3) is the more attractive and useful in the two variable case in its index number application, presently to be discussed. Its main characteristic is its <u>symmetry</u>, with the same coefficients within f_1 and f_2 and between f_1 and f_2. This property of symmetry will be adopted for the solution of (1) which will be found to be unique.

In an index number application x_1 might be price and x_2 quantity, the states time or place, the product value, both for a particular commodity. For many commodities we simply introduce a Σ on both sides of (2), and we have divided the sum product of a set of commodities symmetrically and consistent into the sum of contributions due to price and quantity. One is reminded immediately of I. Fisher's Ideal index number treatment in logarithmic form, except that the present is far simpler and is recommended for trial: the unitary changes in price of quantity between states 1 and 2 are the Σs of the expressions on the right of (2) divided by the Σ value at state 1.

It is hard to envisage economic application for k greater than 2. The problem may be envisaged as a generalisation and without immodesty described as elegant because, as will appear, it is not my own.

In an earlier draft of this paper I gave the solutions for products 2,3 and 4 and announced that there was no symmetrical solution for $k = 5$ or more! I submitted the paper to an economics journal which reasonably hesitated about acceptance because it was not an economics paper. But, from my point of view more importantly, a Reader stated that my conclusion about $k = 5$ or over was wrong - an error in algebra, and very ingeneously suggested a unique general symmetrical solution, i.e. for any number k variables. He emphasized that he did not supply a solution. This is provided here.

In addition to the three properties of f_i announced at the outset, obviously we have (iv) that the coefficients of the products in f_i in (k-1) variables all of the same state must be 1/k (e.g. with $k = 3$ the coefficient of $x_2^2\, x_3^2$ and $x_2^1\, x_3^1$ in f_1 must be 1/3). Other products will be termed mixed (in states, e.g. $x_2^2\, x_3^1$, which must cancel out). In each f_i there will 2^{k-1} product terms but these can be divided binomially into k terms:-

$$(4) \qquad 2^{k-1} = \sum_{n=0}^{k-1} \frac{(k-1)!}{n!\,(k-n-1)!}$$

n may be regarded as the number of state 1 and (k-n-1) the number of state 2 of which there will indeed be

$$(5) \qquad \frac{(k-1)!}{n!\,(k-n-1)!}$$

in every f_i. From symmetry, each of these products must have the same coefficients, say c_n, and also from symmetry the set of coefficients must be the same in all f_i.

As the sum of coefficients in all f_i is unity and the first and last of these is 1/k the Reader had the ingeneous idea of also equating the sum of coefficients of the n - set to 1/k, i.e.

(6) $$c_n \frac{(k-1)!}{n!\,(k-n-1)!} = \frac{1}{k}, \quad n = o, 1, \ldots, (k-1)$$

or

(7) $$c_n = \frac{n!\,(k-n-1)!}{k!}, \quad n = o, 1, \ldots, (k-1)$$

The reader disclaimed proving this; actually proof is easy. From (7), c_o and c_{k-1} are 1/k as we already know. On the right side of (1) there will be equal number of + and - terms. Every term will be in all the variables.

(8) $$x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k},$$

the superscripts a_i being all 1 or 2, permuted in all possible ways. With all the a_i 1 or 2 the coefficients c_o and c_{k-1} are 1/k, as we have seen. On the right of (1) the positive coefficient of any term (8) with n of state 1 will be $(k-n)\,c_n$ and the negative coefficient of the same term is $n\,c_{n-1}$. Hence we must show that

(9) $$(k-n)\,c_n - nc_{n-1} = 0$$

except for n = 0 or n = k - 1. From (7)

(10) $$\frac{c_n}{c_{n-1}} = n!\,\frac{(k-n-1)!}{k!}\;\frac{k!}{(n-1)!\,(k-n)!} = \frac{n}{(k-n)}$$

so that (9) is true.

Following are the values of c_n for number of variables k = 2 - 6 and, in brackets, the number of terms which symmetrically have the same coefficient:-

k 11	n = 0	1	2	3	4	5
2	1/2(1)	1/2(1)				
3	1/3(1)	1/6(2)	1(3(1)			
4	1/4(1)	1/12(3)	1/12(3)	1/4(1)		
5	1/5(1)	1/20(4)	1/30(6)	1/20(4)	1/5(1)	
6	1/6(1)	1/30(5)	1/60(20)	1/60(20)	1/30(5)	1/6(1)

The binomial character of the solution is evident, also that, as symmetrical, is is unique.

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