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A BASIC INPUT-OUTPUT MODEL OF CAPACITY GROWTH, WITH RECURSIVE SOLUTION AND NUMERICAL ILLUSTRATION

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ABSTRACT

A basic input-output model of sector output capacity growth is proposed, having a solution through backward recursion from known capacities of a terminal year T. A non-singular square matrix is found, linking capacity expansion Δx_{T-1} with known capacity differences derived from years T and (T-1). This matrix might be termed the "Capacity Inverse", for its resemblance to the Leontief Inverse of the open static model. Detailed numerical illustration and verification are provided, using Irish 12-sector data and "contrived" structures for which the numerical answers are known, for comparison with model solutions. Some economic interpretations and conclusions are proposed.

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1. *INTRODUCTION AND BACKGROUND*

During 1985-86 the author did input-output (IO) modelling work for the Industrial Development Authority (IDA Ireland) on economic projections, exports, and so on, for the Irish economy. The question arose as to how output capacity of various sectors could be determined consistently for some or all years between a base-year structure and a terminal year for which a full economic structure had been projected and finalised. The problem was therefore expressed in terms of sector output capacities. The paper which follows presents and verifies numerically the basic and rudimentary model solution proposed by the author.

The model solves the problem by backward recursion from terminal year T, fully specified. Required data for each intermediate year t are the A_t matrix, the B_t matrix, and y_t^{non} which is final demand (or final output) not required for capacity expansion. These symbols are described and explained in Section 2 following. The model is, of its nature, similar to Dynamic Input-Output models, but is a rather basic and simple version of such models. There is backward recursion only, without advertence to forward recursion, in the model solution. The emphasis is on sector output capacity, as distinct from output as such. The solution found is very much a particular rather than a general solution. Starting from a definite and complete structure for terminal year T, as first base, the model finds a consistent structure for year (T-1), and then for year

(T-2), using that of year (T-1) as base, and so on backwards to any earlier year (T-k).

The author is aware of a large and growing literature on dynamic IO modelling. Four selected publications of the period 1976-1988 give some indication of the scope and sophistication of the work being done. The Miller and Blair (1985) textbook devotes part of Chapter 9 to dynamic modelling, including problems of instability of solutions in forward recursion, with numerical illustration. Livesey (1976) finds a solution of the dynamic Leontief IO model for matrix B having rows of zero elements, by partitioning and transforming the B-matrix. The Leontief and Duchin (1984) report has a very informative Chapter 2 on the history of the dynamic IO model and various problems with solutions, as well as a methodology of multi-period capital gestation and of allowance for unused capacity, in forward recursion. Szyld (1988) considers the existence of positive solutions to the dynamic IO model when moving forward in time, depending on the initial structure lying on the so-called "balanced growth path".

In the face of so much work, the reader may reasonably ask what the author has to offer, by way of a useful model or application? Two features of the model proposed below may be novel - the author has not seen them in the mainstream literature, as published.

The first feature is the manipulation of the capacity growth formula, whereby capacity expansion Δx_{T-1} can be expressed by application of a robust matrix inverse to a known capacity difference, namely output capacity x_T of year T, less required capacity x_{T-1}^{non} of year (T-1) to provide outputs other than those for capacity expansion. The output capacity x_T is postulated, as the start of backward recursion through year (T-1), year (T-2), and so on.

The second feature is the "Capacity Inverse" solution to the per-unit capacity expansion, an interesting parallel to the Leontief Inverse solution to the per-unit final demand growth of output in the open static model. The term "Capacity Inverse" is proposed by the author as a description of the square matrix showing main characteristics of the usual $(I-A)^{-1}$ Leontief Inverse. The detailed numerical illustration is intended to enable readers to verify results for themselves. The wider implications of the "Capacity Inverse" have not been discussed, to avoid undue length.

The following parts of the paper address major aspects. Section 2 gives algebraic formulation of the model, with description of symbols, and solution for x_{T-1}, given x_T. Section 3 provides a numerical illustration of how the model works, using as core a 12-sector 1982 Irish transactions table. Section 4 offers economic interpretation and conclusions, one conclusion being that the model as proposed is not usable for forward recursion.

2. *THE MODEL, WITH SOLUTION BY BACKWARD RECURSION*

We assume a capacity growth period covering years 1 to T, year T being the terminal year. A year Ø, before the beginning of the growth period, will also have relevance in the discussion of Section 4 below. We may assume valuation at approximate basic values, and at constant prices of some year (say 1980). The basic equation of the Model for typical year t, *at constant prices,* is

$$x_t = A_t x_t + B_t \Delta x_t + y_t^{non} \qquad (2.1)$$

The variables and symbols are as follows:

x_t is the vector of sector gross output capacities of n sectors, during year t.

A_t is the inter-industry direct input coefficient matrix, which might include household income rows and household expenditure columns, of dimension (n, n).

B_t is the matrix of capital flow coefficients of year t, also of dimension (n, n), typically including rows of zero coefficients. The B_t investment is towards capacity expansion for year (t+1), *not* for year t. For capital projects over several years (such as electric power stations), B_t includes those parts (per unit output) completed in year t to permit capacity expansion in year (t+1). Parts constructed in earlier years are included in y_{t-1}^{non}, y_{t-2}^{non}, y_{t-3}^{non}, and so on, in proportion to the value of $B_t \Delta x_t$ when the latter becomes known.

Δx_t is the growth of sector annual output capacity $(x_{t+1} - x_t)$, between year t and year (t+1), of dimension n. At constant prices, x_t, x_{t+1}, Δx_t denote *required capacity or capacity expansion*. Unless otherwise indicated, the system is assumed to work at maximum efficiency of capacity utilisation, meaning that capacity equals output in year t, and is measured by output x_t at constant prices.

y_t^{non} is "exogenous" final demand, *excluding* capital formation in year t to permit capacity expansion Δx_t for year (t+1), but including prior investment such as electric power stations during years before their year of completion. Thus y_t^{non} includes some gross fixed capital formation, as well as exports of goods and services, Government current expenditure, inventory changes, and perhaps some or all of household expenditure on consumers' goods and services. It can also include replacement investment, to counteract scrapping of fixed assets.

All of x, Δx, y, A, B, comprise domestic flows only, for imports excluded. Equation (2.1) says that domestic output x_t supplies inter-industry inputs Ax_t, also supplies "exogenous" final output y_t^{non}, as well as supplying gross fixed capital formation $B\Delta x_t$ to permit capacity expansion Δx_t for year (t+1), available at the end of year t. And Equation (2.1) is to be interpreted as output capacity, as well as output as such, for full capacity utilisation.

Solution of the System by Backward Recursion

A complete solution will now be outlined, to comprise solving terminal year T first, then solving year (T-1), and so on, back to year 1. For each year, a satisfactory solution of the Equation (2.1) system is required.

To solve year T, some feasible or typical growths of sector output capacities might be assumed for year (T+1), such as 1 per cent for agriculture, 10 per cent for engineering. Thus, we may express Δx_T in terms of x_T as

$$\Delta x_T = \hat{\lambda}_T x_T \tag{2.2}$$

$\hat{\lambda}$ being a diagonal matrix of dimension (n, n) having zero entries except for the diagonal. These diagonal locations or elements might have the typical growth rates mentioned, such as 0.01 for Agriculture sector (1) in diagonal location (1,1), 0.10 in Engineering sector (10) diagonal location (10,10), and so on. For no expansion of capacity permitted, $\hat{\lambda}_T$ would be zero everywhere.

In terms of year T and growth rates $\hat{\lambda}_T$, Equation (2.1) becomes

$$(I-A_T - B_T\hat{\lambda}_T)x_T = y_T^{non} \tag{2.3}$$

The solution, in typical "Leontief-Inverse" form is

$$x_T = (I-A_T - B_T\hat{\lambda}_T)^{-1} y_T^{non} \tag{2.4}$$

where I is the unit matrix of dimension (n,n). To solve for year (T-1), an algebraic ruse is required. We first look at basic Equation (2.1) for year (T-1):

$$x_{T-1} = A_{T-1}\, x_{T-1} + B_{T-1}\, \Delta x_{T-1} + y_{T-1}^{non} \tag{2.5}$$

The algebraic manipulation has two components:

(a) Express Δx_{T-1}, as $\hat{\lambda}_{T-1} x_\emptyset$, thus free of x_T and x_{T-1}, for $\hat{\lambda}_{T-1}$ elements unknown but $x_\emptyset$ known.

(b) Replace the left-hand-side (LHS) of (2.8) below, x_{T-1} by $x_T - \Delta x_{T-1}$ ($= x_T - x_T + x_{T-1} = x_{T-1}$), thus the LHS of

(2.8) below becomes $x_T - \hat{\lambda}_{T-1} x_\emptyset$.

Making these substitutions leads to

$$x_{T-1} = A_{T-1}x_{T-1} + B_{T-1}\hat{\lambda}_{T-1} x_\emptyset + y_{T-1}^{non} \quad (2.6)$$

$$(I-A_{T-1})x_{T-1} = B_{T-1}\hat{\lambda}_{T-1} x_\emptyset + y_{T-1}^{non} \quad (2.7)$$

$$x_{T-1} = (I-A_{T-1})^{-1} (B_{T-1}\hat{\lambda}_{T-1} x_\emptyset + y_{T-1}^{non}) \quad (2.8)$$

Now we modify the LHS of (2.8):

$$x_T - \hat{\lambda}_{T-1}x_\emptyset = (I-A_{T-1})^{-1}B_{T-1}\hat{\lambda}_{T-1}x_\emptyset + (I-A_{T-1})^{-1}y_{T-1}^{non} \quad (2.9)$$

This, rearranged, gives

$$x_T - (I-A_{T-1})^{-1}y_{T-1}^{non} = \left[I + (I-A_{T-1})^{-1}B_{T-1}\right]\hat{\lambda}_{T-1}x_\emptyset \quad (2.1\emptyset)$$

with all the LHS known.

Thus $\hat{\lambda}_{T-1}x_\emptyset = \left[I + (I-A_{T-1})^{-1}B_{T-1}\right]^{-1}$ (LHS of (2.1∅)) (2.11)

It follows that

$$x_{T-1} = x_T - \hat{\lambda}_{T-1}x_\emptyset \quad (2.12)$$

The LHS of (2.11) is, of course, Δx_{T-1}. This needs to be subtracted from x_T, to give x_{T-1}, per Equation (2.12). The first term of the right-hand-side (RHS) of (2.11) is likely to have an inverse, because of I forming the diagonal, apart from generally small entries elsewhere. The numerical illustration below shows that no problems of inversion need occur. The form of the first terms on the RHS of (2.1∅) and (2.11) will be considered in Section 4. below.

By further recursion, one can move back to year (T-2), and so on. There is no condition of constant technology imposed on the A and B matrices, which may change from year to year, if required to. Once Δx_{T-1} is known, the prior investment entries for power stations, etc., can be entered in y^{non}_{T-2}, y^{non}_{T-3}, and so on.

The solutions thus obtained for x_T and x_{T-1} have a clear meaning, provided that sector outputs generally increase from year (T-1) to year T to year (T+1). The capacity expansion by way of $b\Delta x_{T-1}$ and $B\Delta x_T$ has then an unambiguous meaning. But if any elements of, say, Δx_{T-1} are negative, then the solution Transactions Table structure includes ***negative*** GFCF columns, which imply full redistribution of spare capacity so that no idle capacity occurs. One solution of this difficulty is to replace such negative elements of $B\Delta x_{T-1}$ by zero, meaning we carry excess capacity into year T. The revised transactions table for year (T-1) has corresponding ***zero*** GFCF columns, to give a revised and larger value of vector x_{T-1}. But this means revised and smaller Δx_{T-1}, for x_T constant; thus some iteration is required before a final Transactions structure for year (T-1) emerges. This aspect is illustrated below in the numerical examples, which provide numerical verification of the model for Δx_t all positive, and for Δx_t having both positive and negative elements.

3. *NUMERICAL ILLUSTRATION OF THE MODEL, USING IRISH 1982 12-SECTOR DATA*

Numerical testing and verification of the Model of Section 2 are now described, in what follows. The core of the data base comprises a 12-sector 1982 Irish input-output (IO) structure derived from Table 5.6 of Henry (1986). From contrived total final demands for 1981 and 1983, at 1982 approximate basic values, sector outputs for 1981 and 1983 are derived by means of the 1982 Leontief Inverse $(I-A_{82})^{-1}$. These total final demands are contrived so as to give Δx_{81} having positive and negative elements, whereas Δx_{82} has positive elements only.

We therefore know the answers in advance. We know x_{81}, x_{82}, x_{83}, so we have Δx_{81} and Δx_{82}. By means of a B-matrix applied to Δx_{81} and Δx_{82} we can get the aggregate gross fixed capital formation (GFCF) part of final demand required for capacity expansion. This aggregate GFCF for 1981 is the *net* result of positive and negative capacity expansion, through positive and negative Δx_{81} elements, respectively. For 1982, all capacity expansion is positive, since all elements of Δx_{82} are positive. The non-capacity final demand shares y_{81}^{non} and y_{82}^{non} comprise total final demand less (net) aggregate GFCF for capacity expansion.

From this available information, we ask the model to estimate Δx_{81} and Δx_{82}, so that we may compare them with the actual values. The full data set is shown, as Tables 1 to 8, to enable readers to verify results on their own computers, if they wish. In view of so much tabular material on display, a minimum of verbal description is required. The following discussion first looks at the build-up towards the Δx estimates, covering Tables 1-7; then the Δx estimates themselves are considered as appearing in Table 8.

Data Preparations (Tables 1 to 7)

Table 1 shows Irish 1982 12-sector transactions, at 1982 approximate basic values. Domestic flows (excluding all

Table 1: Irish 1982 12-Sector Transactions at Approximate Basic Values, £million, current

Sectors	Energy (1)	Agriculture forestry fishing (2)	Food drink tobacco (3)	Clothing, footwear, textiles (4)	Wood, paper, Misc. Manufact. (5)	Chemicals (6)	Non-metallic minerals + mining (7)	Engineering (8)	Construction (9)	Transport (10)	Commerce (11)	Public + professional (12)	Total Inter	Total Final	Total Output
Energy (1)	155.5	46.0	39.0	11.0	13.0	26.0	32.0	31.0	10.0	11.0	42.3	34.7	451.5	467.0	918.5
Ag, for, fish, (2)		24.0	1,464.2		2.1						1.1	0.9	1,492.3	712.7	2,205.0
Food, drink, tob. (3)		247.0	824.0	28.2							9.8	8.2	1,117.2	3,021.8	4,139.0
Cloth., footw. text (4)				23.2	0.7								23.9	585.1	609.0
Wood, paper, misc., (5)					92.1				29.0		10.9	9.1	141.1	745.9	887.0
Chemicals (6)		58.0	3.6	2.5		56.8		43.3			2.8	2.2	169.2	988.8	1,158.0
Non-metallic minerals + mining (7)		9.0					98.6		332.9		1.1	0.9	442.5	239.5	682.0
Engineering (8)	2.0	51.0		5.7	2.3	9.2	5.5	128.9	116.0	50.0	12.6	10.4	393.6	2,327.4	2,721.0
Construction (9)	7.0		17.0				4.0		317.1	17.0	10.9	9.1	382.1	1,981.9	2,364.0
Transport (10)		44.0							109.0		13.3	3.7	170.0	662.0	832.0
Commerce (11)	4.6	292.3	88.0	39.4	21.5	47.6	61.1	175.1	135.6	2.7	25.4	9.7	903.0	2,928.0	3,831.0
Public + Prof. (12)	0.4	33.7	10.0	4.6	2.5	5.4	6.9	19.9	15.4	0.3	2.1	1.8	103.0	2,292.0	2,395.0
TOTAL INTER	169.5	805.0	2,445.8	114.6	134.2	145.0	208.1	398.2	1,065.0	81.0	132.3	90.7	5,789.4	16,952.1	22,741.5
TOTAL INPUT	918.5	2,205.0	4,139.0	609.0	887.0	1,158.0	682.0	2,721.0	2,364.0	832.0	3,831.0	2,395.0			22,741.5

Source: Table 5.6 of Henry (1986).

Table 2: *Leontief Inverse $(I-A_{82})^{-1}$ derived from Table 1 12-Sector Structure*

Sectors		Energy (1)	Agric-ulture (2)	Food (3)	Cloth-ing (4)	Wood (5)	Chem-icals (6)	Non-Metall. (7)	Engin-eering (8)	Constr-uction (9)	Trans-port (10)	Comm-erce (11)	Public (12)
Energy	(1)	1.20412	.03260	.02912	.02536	.02028	.02925	.06799	.01596	.02010	.01734	.01375	.01793
Agric.	(2)	.00001	1.06450	.47023	.02276	.00288	.00008	.00019	.00012	.00020	.00002	.00153	.00203
Food	(3)	.00003	.14943	1.31468	.06356	.00056	.00017	.00042	.00027	.00036	.00004	.00344	.00458
Clothing	(4)	0.	0.	0.	1.03960	.00092	0.	0.	0.	.00001	0.	0.	0.
Wood	(5)	.00017	.00058	.00044	.00028	1.11597	.00017	.00051	.00026	.01618	.00036	.00325	.00432
Chemicals	(6)	.00006	.03019	.01451	.00543	.00016	1.05177	.00028	.01764	.00117	.00109	.00089	.00112
Non-metall.	(7)	.00176	.00549	.00345	.00027	.00007	.00009	1.17053	.00010	.19067	.00393	.00093	.00122
Engineering	(8)	.00338	.02830	.01301	.01126	.00330	.00905	.01101	1.05023	.06511	.06450	.00399	.00506
Constr.	(9)	.01065	.00212	.00711	.00082	.00029	.00043	.00895	.00041	1.15799	.02384	.00355	.00464
Transport	(10)	.00052	.02188	.01005	.00076	.00017	.00018	.00081	.00027	.05372	1.00113	.00369	.00182
Commerce	(11)	.00708	.14927	.09329	.07330	.02805	.04433	.10725	.06894	.08908	.00933	1.00770	.00540
Public	(12)	.00064	.01717	.01068	.00853	.00325	.00502	.01209	.00782	.01009	.00105	.00667	1.00090

Table 3: *Sector Outputs x and Changes* Δ x
£million at 1982 prices

Sectors		1981 x_{e1} (1)	1982 x_{e2} (2)	1983 x_{e3} (3)	$\Delta x_{e1} = x_{e2} - x_{e1}$ (4)	$\Delta x_{e2} = x_{e3} - x_{e2}$ (5)
Energy	(1)	917.214	918.5	933.827	1.286	15.327
Agriculture	(2)	2,081.084	2,205.0	2,328.983	123.916	123.983
Food	(3)	3,877.807	4,139.0	4,400.346	261.193	261.346
Clothing	(4)	559.492	609.0	658.508	49.508	49.508
Wood	(5)	859.507	887.0	914.643	27.493	27.643
Chemicals	(6)	1,254.661	1,158.0	1,262.603	-96.661	104.603
Non-Metall.	(7)	701.899	682.0	703.590	-19.899	21.590
Engineering	(8)	2,923.057	2,721.0	2,931.015	-202.057	210.015
Construction	(9)	2,362.797	2,364.0	2,365.760	1.203	1.760
Transport	(10)	829.445	832.0	834.723	2.555	2.723
Commerce	(11)	3,823.774	3,831.0	3,877.128	7.226	46.128
Public	(12)	2,394.140	2,395.0	2,400.269	.860	5.269
TOTAL		22,584.877	22,741.5	23,611.395	156.623	869.895

imports) are shown, as well as total intermediate output, total final demand, and total output, same as total input. The total inter-industry input shows very small shares of total input for sectors (10) to (12), a possible source of "noise" in the system of twofold matrix inversion, for which single precision was used, as part of the test.

Table 2 shows the Leontief Inverse $(I-A_{82})^{-1}$ derived from Table 1 12-sector inter-industry transactions.

Table 3 shows sector outputs x for 1981, 1982 and 1983, as well as derived Δx_{81} and Δx_{82}. We see that sectors (6) to (8) have negative elements in Δx_{81}, whereas all the elements of Δx_{82} are positive. The sector outputs x_{81} and x_{83} were obtained by post-multiplying Table 2 Leontief Inverse by the total final demands of 1981 and 1983, shown in Table 7 columns (3) and (7), respectively. All outputs are to be thought of as capacities.

Table 4 provides the data for the B-matrix and derived GFCF of capacity expansion during 1981 and 1982. The four domestic non-zero rows of the B-matrix occupy the upper portion of the Table. The coefficients for imported capital goods, per £ unit of sector output capacity, also appear, although not used by the model. The B-matrix therefore has eight zero rows. The basic data for the capital coefficients appear in a Henry (1989) forthcoming study of Domestic Wealth in Ireland. We see very large coefficients for Construction row (9) capital input per unit of capacity output. This row (9) might be spread over several years (re. electric power

Table 4: B-Matrix and GFCF* Domestic for Capacity Expansion Δx_{81} and Δx_{82}, excluding Capital Goods Imported

Item	Energy (1)	Agriculture (2)	Food (3)	Clothing (4)	Wood (5)	Chemicals (6)	Non-Metallic (7)	Engineering (8)	Construction (9)	Transport (10)	Commerce (11)	Public (12)	Positive Aggreg. Domestic GFCF for Capacity Change (13)	Negative Aggreg. Domestic GFCF for Capacity Change (14)	Net Aggreg. Domestic GFCF for Capacity Change (15)
B-Matrix Non-Zero rows															
Engineering (8)	.3308	.4450	.1624	.3350	.2892	.2321	.5113	.1436	.1667	.5409	.2902	.3552			
Construction (9)	2.4227	1.0776	.2904	.7223	.4948	.3172	.6534	.2800	.0641	1.6056	1.1884	2.2013			
Transport (10)	0.0	.0309	.0113	.0233	.0201	.0161	.0355	.0100	.0116	0.0	0.0	.0228			
Commerce (11)	0.0	.0927	.0338	.0698	.0602	.0484	.1065	.0299	.0347	0.0	0.0	0.0			
Imports	.4961	.6676	.2436	.5023	.4338	.3481	.7668	.2154	.2500	.8113	.4353	.5329			
TOTAL Capital/Output Coefficient	3.2496	2.3138	.7415	1.6527	1.2981	.9619	2.0735	.6789	.5271	2.9578	1.9139	3.1122			
Domestic GFCF for Capacity Change Δx_{81}															
Engineering (8)	.425	55.143	42.418	16.585	7.951	-22.435	-10.174	-29.015	.201	1.382	2.097	.305	126.507	-61.624	64.883
Construction (9)	3.116	133.532	75.850	35.760	13.604	-30.661	-13.002	-56.576	.077	4.102	8.587	1.893	276.521	-100.239	176.282
Transport (10)	0.0	3.829	2.951	1.154	.553	-1.556	-.706	-2.021	.014	0.0	0.0	.020	8.521	-4.283	4.238
Commerce (11)	0.0	11.487	8.828	3.456	1.655	-4.678	-2.119	-6.042	.042	0.0	0.0	0.0	25.468	-12.839	12.629
Total Domestic GFCF	3.541	203.991	130.047	56.955	23.763	-59.330	-26.001	-93.654	.334	5.484	10.684	2.218	437.017	-178.985	258.032
Δx_{81} itself	1.286	123.916	261.193	49.508	27.493	-96.661	-19.899	-202.057	1.203	2.555	7.226	.860			
Domestic GFCF for Capacity Change Δx_{82}															
Engineering (8)	5.070	55.172	42.443	16.585	7.994	24.278	11.039	30.158	.293	1.473	13.386	1.872	209.763		209.763
Construction (9)	37.133	133.604	75.695	35.760	13.678	33.180	14.107	58.804	.113	4.372	54.819	11.599	473.064		473.064
Transport (10)	0.0	3.831	2.953	1.154	.556	1.684	.766	2.100	.020	0.0	0.0	.120	13.184		13.184
Commerce (11)	0.0	11.493	8.833	3.456	1.664	5.063	2.299	6.279	.061	0.0	0.0	0.0	39.148		39.148
Total Domestic GFCF	42.203	204.100	130.124	56.955	23.892	64.205	28.211	97.341	.487	5.845	68.205	13.591	735.159	Nil	735.159
Δx_{82} itself	15.327	123.983	261.346	49.508	27.643	104.603	21.590	210.015	1.760	2.723	46.128	5.269			

* GFCF is Gross Fixed Capital Formation

Table 5: the $[I+(I-A_{82})^{-1}B_{82}]$ Matrix, referred to in the text as the $(I-C)^{-1}$ Capacity Inverse

Sectors		Energy (1)	Agriculture (2)	Food (3)	Clothing (4)	Wood (5)	Chemicals (6)	Non-Metall. (7)	Engineering (8)	Construction (9)	Transport (10)	Commerce (11)	Public (12)
Energy	(1)	1.05398	.03057	.00909	.02123	.01574	.01102	.02337	.00850	.00463	.04090	.02852	.05031
Agricult.	(2)	.00051	1.00041	.00013	.00029	.00022	.00016	.00035	.00012	.00009	.00038	.00027	.00047
Food	(3)	.00095	.00082	1.00026	.00059	.00046	.00034	.00074	.00024	.00019	.00072	.00050	.00088
Clothing	(4)	.00003	.00002	0.	1.00001	.00001	0.	.00001	0.	0.	.00002	.00002	.00003
Wood	(5)	.03928	.01786	.00485	.01201	1.00828	.00535	.01106	.00467	.00120	.02611	.01930	.03571
Chemicals	(6)	.00868	.00923	.00325	.00684	.00576	1.00453	.00992	.00290	.00306	.01143	.00651	.00887
Non-Metall.	(7)	.46195	.20571	.05546	.13791	.09450	.06061	1.12487	.05347	.01232	.30618	.22661	.41983
Engin.	(8)	.50516	.53988	.19033	.40064	.33748	.26564	.58224	1.16981	.18013	.67261	.38215	.51784
Constr.	(9)	2.80560	1.24910	.33674	.83736	.57379	.36780	.75807	.32464	1.07470	1.85949	1.37628	2.54978
Transport	(10)	.13024	.08929	.02708	.06248	.04701	.03340	.07118	.02520	.01523	1.08640	.06392	.14118
Commerce	(11)	.23861	.22037	.07123	.15799	.12486	.09318	.20110	.06506	.05228	.18031	1.12586	.22078
Public	(12)	.02703	.01445	.00423	.00998	.00732	.00506	.01070	.00398	.00199	.02043	.01426	1.02501

Table 6: the $[I+(I-A_{82})^{-1}B_{82}]^{-1}$ Matrix, referred to in the text as the (I-C) Capacity Matrix

Sectors		Energy (1)	Agric-ulture (2)	Food (3)	Clothing (4)	Wood (5)	Chem-icals (6)	Non-Metall. (7)	Engin-eering (8)	Constr-uction (9)	Trans-port (10)	Comm-erce (11)	Public (12)
Energy	(1)	.96926	-.01595	-.00465	-.01099	-.00802	-.00553	-.01166	-.00438	-.00210	-.02341	-.01629	-.02829
Agricult.	(2)	-.00016	.99979	-.00007	-.00016	-.00013	-.00010	-.00021	-.00007	-.00006	-.00012	-.00009	-.00015
Food	(3)	-.00021	-.00043	.99985	-.00032	-.00026	-.00021	-.00045	-.00013	-.00014	-.00018	-.00012	-.00020
Clothing	(4)	-.00024	-.00001	0.	1.0	0.	0.	0.	0.	0.	-.00002	-.00001	-.00002
Wood	(5)	-.02921	-.01020	-.00243	-.00656	.99596	-.00223	-.00435	-.00244	.00046*	-.01808	-.01382	-.02590
Chemicals	(6)	.00237*	-.00366	-.00166	-.00304	-.00303	.99731	-.00606	-.00140	-.00245	-.00361	-.00090	.00133*
Non-Metall.	(7)	-.34905	-.11789	-.02752	-.07533	-.04564	-.02457	.95270	-.02783	.00704*	-.21556	-.16495	-.30942
Engin.	(8)	.14430*	-.21256	-.09719	-.01770	-.17735	-.15758	-.35578	.91849	-.14441	-.21348	-.05216	.08220*
Constr.	(9)	-2.12023	-.71589	-.16706	-.45742	-.27705	-.14913	-.28704	-.16896	1.04282	-1.30934	-1.00193	-1.87950
Transport	(10)	-.06144	-.04926	-.01513	-.03464	-.02631	-.01885	-.04030	-.01404	-.00902	.96518	-.02779	-.07679
Commerce	(11)	-.03681	-.11489	-.04072	-.08549	-.07227	-.05717	-.12531	-.03629	-.03913	-.03321	.97844	-.03429
Public	(12)	-.01643	-.00755	-.00213	-.00514	-.00365	-.00244	-.00509	-.00202	-.00074	-.01239	-.00866	.98509

* There are 6 positive entries among the 132 "expected" negative entries.

station construction, and so on), but is all loaded into the one year, as a test of the stability of the structure, under matrix inversion. The same B-matrix is used for 1981 and 1982 capacity expansion. Small shares of GFCF, such as furniture, etc., have been ignored, with coefficients confined to construction, engineering goods, and trade and transport margins on the latter.

The middle section of Table 4 shows GFCF results of applying B-matrix coefficients to Δx_{81}. Cumulated positive and negative results appear in columns (13) to (15), the net aggregate being in column (15), some £258m. of GFCF, the major share of which comprises £176m. of Construction output sector (9). The lower section of Table 4 shows corresponding GFCF results for Δx_{82}, all positive, and aggregating to about £735m. of which £473m. is Construction output. Deduction of these GFCF amounts from Total Final Demand provides y_{81}^{non} and y_{82}^{non}, as shown in Table 7.

Table 5 shows the 12-sector $[I + I-A_{82})^{-1}B_{82}]$ matrix derived from Tables 2 and 4 above, with addition of the unit matrix I. The Table 5 matrix shows all positive entries, with near-unit values in the diagonal. Some rather large entries appear in the Construction row (9), but otherwise most off-diagonal entries are smaller than unity. Because of its similarity to the Leontief Inverse $(I-A)^{-1}$ structure, this writer tentatively suggests the description "$(I-C)^{-1}$ Capacity Inverse", to be commented on in Section 4 below. The Table 5 matrix is used for 1981 and 1982 calculations, as a constant structure.

Table 6 shows the inverse of Table 5. The Table 6 matrix shows near-unity positive values on the diagonal, with off-diagonal values generally negative and less than unity. Some large negative entries appear in Construction row (9). Six off-diagonal positive entries occur, and are marked by an asterisk (*). This Table 6 matrix bears a strong resemblance of form to the Leontief (I-A), so this writer tentatively suggests the description "(I-C) Capacity Matrix", the counterpart of the suggested inverse form of Table 5.

In Table 7 there is shown the breakdown of 1981 and 1982 final demands between GFCF for capacity expansion and y^{non}. The GFCF referred to has appeared already in Table 4, with comment given above. The detailed breakdown of total final demand of 1981 and 1982 occupies columns (1) to (6) of Table 7. Column (7) shows 1983 total final demand, for which no breakdown is required by the modelling exercise. Columns (8) and (9) of Table 7 show partial sector outputs of 1981 and of 1982, respectively, obtained by postmultiplying $(I-A)^{-1}$ by y^{non} for both years. These partial outputs are required by the LHS of Equation (2-10) above.

The Δx *Estimates Compared with Actuals* (Table 8)

The final preparatory data need to be considered first. These occupy columns (1) and (4) of Table 8, and correspond to the LHS of Equation (2.10) above. They comprise the sector output values of Table 3 columns (2) and (3), less values of Table 7 columns (8) and (9), respectively, to give one LHS of

Table 7: Final Demands, and $(I-A)^{-1}y^{non}$, for 1981, 1982 and 1983
£ million at 1982 prices

Sectors	Final Demands: 1981 y^{non}_{81}	Final Demands: 1981 Net Aggreg. Domestic GFCF for Capacity Change	Final Demands: 1981 TOTAL FINAL (1)+(2)	Final Demands: 1982 y^{non}_{82}	Final Demands: 1982 Net Aggreg. Domestic GFCF for Capacity Change	Final Demands: 1982 TOTAL FINAL (4)+(5)	Final Demands: 1983 TOTAL FINAL	Partial Sector Outputs: 1981 $(I-A_{81})^{-1}y^{non}_{81}$	Partial Sector Outputs: 1982 $(I-A_{82})^{-1}y^{non}_{82}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Energy (1)	467.0		467.0	467.0		467.0	467.0	912.388	904.878
Agricult. (2)	682.6		682.6	712.7		712.7	742.8	2,081.022	2,204.822
Food (3)	2,828.8		2,828.8	3,021.8		3,021.8	3,214.8	3,877.683	4,138.640
Clothing (4)	537.5		537.5	585.1		585.1	632.7	559.490	608.993
Wood (5)	721.3		721.3	745.9		745.9	770.5	856.597	879.161
Chemicals (6)	1,081.2		1,081.2	988.8		988.8	1,081.2	1,253.293	1,153.696
Non-Metall. (7)	257.2		257.2	239.5		239.5	257.2	668.253	591.695
Engineering (8)	2,457.717	64.883	2,522.6	2,117.637	209.763	2,327.4	2,522.6	2,843.114	2,468.894
Constr. (9)	1,805.618	176.282	1,981.9	1,508.836	473.064	1,981.9	1,981.9	2,158.491	1,815.657
Transport (10)	657.762	4.238	662.0	648.816	13.184	662.0	662.0	815.667	793.185
Commerce (11)	2,915.371	12.629	2,928.0	2,888.852	39.148	2,928.0	2,928.0	3,790.831	3,734.829
Public (12)	2,292.0		2,292.0	2,292.0		2,292.0	2,292.0	2,391.842	2,388.547
TOTAL	16,704.068	258.032	16,962.1	16,216.941	735.159	16,952.1	17,552.7	22,208.671	21,682.997

Table 8: Model Results for Δx_{81} and Δx_{82} compared with actual values, and Model iterative results for x_{81}, at 1982 prices.
£ million

Sectors	for Δx_{81}			for Δx_{82}			Further x_{81} results		
	x_{82} less $(I-A_{81})^{-1}y^{non}_{81}$	Model estimate Δx_{81} based on (1)	Actual Δx_{81}	x_{83} less $(I-A_{82})^{-1}y^{non}_{82}$	Model estimate Δx_{82} based on (4)	Actual Δx_{82}	x_{81} original from net GFCF plus y^{non}_{81}	x_{81} from positive GFCF plus y^{non}_{81}	x_{81} after 4 iterations
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Energy (1)	6.112	1.286	1.286	28.949	15.327	15.327	917.214	920.461	920.010
Agriculture (2)	123.978	123.916	123.916	124.161	123.983	123.983	2,081.084	2,081.131	2,081.126
Food (3)	261.317	261.193	261.193	261.706	261.346	261.706	3,877.807	3,877.903	3,877.895
Clothing (4)	49.510	49.508	49.508	49.515	49.509	49.508	559.492	559.494	559.493
Wood (5)	30.403	27.492	27.493	35.482	27.643	27.643	859.507	861.189	860.887
Chemicals (6)	-95.293	-96.660	-96.661	108.907	-104.603	104.603	1,254.661	1,255.881	1,255.774
Non-Metall. (7)	13.747	-19.899	-19.899	111.895	21.591	21.590	701.899	721.044	717.511
Engineering (8)	-122.114	-202.057	-202.057	462.121	210.013	210.015	2,923.057	2,994.625	2,988.333
Construction (9)	205.509	1.199	1.203	550.103	1.764	1.760	2,362.797	2,479.043	2,457.583
Transport (10)	16.333	2.557	2.555	41.538	2.723	2.723	829.445	839.091	838.120
Commerce (11)	40.169	7.228	7.226	142.299	46.126	46.128	3,823.774	3,849.926	3,847.809
Public (12)	3.158	.859	.860	11.722	5.269	5.269	2,394.140	2,395.647	2,395.422
TOTAL	532.829	156.622	156.623	1,928.398	869.897	869.895	22,584.877	22,835.435	22,799.963

(2.10) for 1981 and one LHS for 1982. We may notice that the LHS values of Table 8 column (1) have two negative entries, for sectors (6) and (8).

In accord with Equation (2.11) above, the LHS values of columns (1) and (4) should be pre-multiplied by the Table 6 matrix, to give Δx estimates. These estimates appear in Table 8 columns (2) and (5), matched by Δx actual values of columns (3) and (6), respectively.

The Δx_{81} results may be considered first. The estimates in column (2) are very close to the actuals in column (3), for all 12 sectors, including the three negative entries. The Δx_{82} outcome for 1982 is equally satisfactory, as shown by comparing columns (5) and (6). Agreement of the estimates with the actuals is quite close, for all 12 sectors. It is apparent, therefore, that the model developed in Section 2 is operable, and gives usable results. The numerical testing has verified it. The results do not show any "noise" occurring in the sets of solutions.

Columns (7) to (9) of Table 8 show further 1981 results, related to the iterative solution given in column (9). Column (7) repeats the 1981 sector output capacity results already shown in Table 3 column (1), resulting from final demands y_{81}^{non} plus *net* GFCF capacity-building. Column (8) shows larger 1981 values, resulting from y_{81}^{non} plus ***gross positive*** GFCF capacity-building, the latter GFCF aggregate £437.017m. appearing in Table 4 column (13). This column (8) result implies that capacity substitution between sectors is *not* allowed for 1982 versus 1981. For 1981 outputs greater than those of corresponding 1982 sectors, the 1981 excess capacity is held idle during 1982. Thus, per Table 4 data, the 1981 GFCF of £437m. is required as gross positive capacity-building, rather than the net £258m. This larger 1981 GFCF effect explains 1981 output capacities in Table 8 column (8) larger than those shown in column (7), in aggregate some £250m. larger.

The Iterative procedure now asks whether any 1981-82 positive Δx_{81} elements derived from column (8) are smaller than those derived from column (7)? If so, then 1981 GFCF capacity-building should be reduced, and so on. In fact, there is little scope for iterative manoeuvre, because eight of the 12 column (8) values exceed those of 1982 (Table 3 column (2)), the four exceptions being sectors (2), (3), (4) and (5). After four iterations, the stable result emerges, as given in Table 8 column (9). We see an aggregate £35m. reduction of capacity, by comparison with the aggregate of column (8), capacity being measured by sector outputs at 1982 prices.

4. *ECONOMIC INTERPRETATION AND CONCLUSIONS*

Equation (2.10) is the key equation to the solution for x_{T-1}, given x_T, in backward recursion. In the light of what has been said in Sections 2 and 3 above, Equation (2.10) can be reformulated for the general year t:

$$x_{t+1} - x_t^{non} = (I-C_t)^{-1}\Delta x_t \qquad (4.1)$$

where

x_{t+1} is the capacity required at the beginning of year (t+1), assumed given.

x_t^{non} is the direct plus indirect capacity required to satisfy y_t^{non}, and given by

$$x_t^{non} = (I-A_t)^{-1} y_t^{non} \qquad (4.2)$$

as indicated in (2.10) above.

$(I-C_t)^{-1}$ is the "Capacity Inverse", indicating the direct plus indirect capacity (increase) required per unit increase of Δx_t, for interpretation of $(I-C_t)^{-1}$ as an inverse.

Δx_t is the increase in capacity required to be available at the beginning of year (t+1), and to be made available by the GFCF of year t devoted to $B\Delta t$. The solution Δx_t to the inverse Equation (2.11) may have negative elements, as illustrated in the numerical examples.

Equation (4.2) is the Leontief Inverse traditional solution, whereby y_t yields x_t as required sector outputs, through the Leontief Inverse $(I-A_t)^{-1}$. Equation (4.1) may be interpreted in parallel. A unit of any element j of Δx_t capacity expansion requires direct-plus-indirect capacity amounts in all sectors, as indicated by column j of $(I-C)^{-1}$. The full requirement for Δx_t is given by the LHS of (4.~~2~~1). This capacity requirement is the total capacity x_{t+1} required for year (t+1), less the capacity x_t^{non} required in year t to satisfy y_t^{non} final demand other than capacity-building GFCF. Equation (4.1) indicates that for $\Delta x_t = 0$, so is the LHS, meaning $x_{t+1} = x_t^{non}$, which makes sense.

Conclusions

Three tentative conclusions are offered:

(1) In backward recursion starting with x_T, the earliest set of capacities x_1 derived as part of the series may differ considerably in structure from the $x_\emptyset$ of year $\emptyset$, not a part of the series. For capacity and capital stock fixed within sectors, rather than freely saleable or rentable for all sectors, there may therefore be a considerable discrepancy between the available $x_\emptyset$ and the required x_1.

(2) In backward recursion from x_{t+1} given, to find x_t by Δx_t and the inverse solution (2.11), we have seen that Δx_t may have positive and negative capacity elements. This was mentioned at the end of Section 2 above. In the iterative solution required for replacement of negative elements of Δx_t by zero, the Model of Section 2 is not needed. The iterations are performed at the Transactions Table level, on repeated values of $B\Delta x_t$, for x_{t+1} a given constant vector, and all elements of Δx_t positive or zero. Repeated applications of the system

$$x_t = (I-A_t)^{-1} (y_t^{non} + B\Delta x_t) \qquad (4.3)$$

can be made, for Δx_t tending towards a constant vector, as it is re-estimated repeatedly as $(x_{t+1} - x_t)$. This process has been verified as operable, in the results shown in Table 8 column (9).

(3) *Forward Recursion* may be considered very briefly. Equation (4.1) sets the picture. Given Δx_t and x_t^{non}, x_{t+1} emerges as the solution. *But*, to know Δx_t, one must know x_{t+1} also. In this case, Equation (4.1) is tautological.

One may ask how could one really estimate x_{t+1} without explicit reference to y_{t+1}^{non}, which forms the core of the Leontief Inverse approach to sector output (or capacity) solutions? However, a forward approach through a speculative Δx_t may be considered. Equation (4.3) will give x_t, which with Δx_t gives x_{t+1}, anyway, regardless of Equation (4.1). But this latter x_{t+1} vector of capacities has no explicit link with any required or actual x_{t+1}^{non}, leaving aside any capacity expansion in year (t+1) itself. It may be concluded that the Model system described above does not enable forward recursion to be made satisfactorily, mainly because of no link-up with x_{t+1}^{non}.

5. *REFERENCES*

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