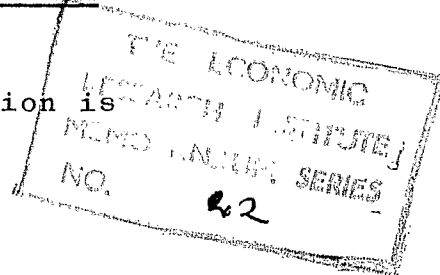


Most Efficient Least Squares Estimator in
Multivariate Regression



The model in matrix notation is

$$(1) \quad \begin{matrix} y & = & X & \beta & + & u \\ (T1) & & (Tk) & (k1) & & (T1) \end{matrix}$$

with X standardized, i.e. $X'X = T1_k$. The regression coefficient estimator is

$$(2) \quad b = (X'X)^{-1}X'y = \frac{1}{T}X'y = \beta + \frac{1}{T}X'u.$$

Let

$$(3) \quad b_1 = \frac{1}{T}R'y$$

be any linear estimator of β . The object is to show that the minimum value of δ , where

$$(4) \quad \delta = E(y_1 - \eta)'(y_1 - \eta)$$

with

$$(5) \quad y_1 = Xb_1; \quad \eta = X\beta$$

is attained for $R = X$.

From (1), (3) and (5),

$$(6) \quad y_1 = \frac{1}{T}XR'(X\beta + u).$$

Hence

$$(7) \quad y_1 - \eta = -X\beta + \frac{1}{T}XR'X\beta + \frac{1}{T}XR'u,$$

so that, from (4),

$$(8) \quad \varphi = -\beta'X'R\beta - \beta'R'X\beta + \frac{1}{T}\beta'X'RR'X\beta + \frac{1}{T}Eu'RR'u,$$

where φ is the value of the terms in R in (4).

We now propose finding the minimum value of φ from

$$(9) \quad \frac{\partial \varphi}{\partial r_{ti}} = 0, \quad t = 1, 2, \dots, T; \quad i = 1, 2, \dots, k.$$

or, rather, showing that (9) is satisfied by $r_{ti} = x_{ti}$, these being the respective elements of R and X . Call the terms on the r.s. of (8) T_1, T_2, T_3, T_4 . Clearly, w.l.g., T_4 , which equals $\sigma^2 \text{Tr} R R'$, can be taken as constant independent of R , e.g. $= \sigma^2$. Of course, $T_1 = T_2$. Set

$$(10) \quad X = Z = \{z_1, z_2, \dots, z_T\}$$

it can then easily be shown that

$$(11) \quad \frac{\partial T_1}{\partial r_{ti}} = -z_{ti} = \frac{\partial T_2}{\partial r_{ti}}$$

and not quite so easily that

$$(12) \quad \frac{\partial T_3}{\partial r_{ti}} = \frac{2}{T} z_t \left(\sum_{s=1}^T z_s r_{si} \right)$$

with, from (10),

$$(13) \quad z_t = \sum_{j=1}^k x_{sj} \beta_j.$$

On substitution for z_s , given by (13) in the brackets () in (12), setting $r_{si} = x_{si}$ and using the orthogonal property of X , we find

$$(14) \quad \frac{\partial T_3}{\partial r_{ti}} = 2 z_t \beta_i.$$

Accordingly, from (11) and (14),

$$(15) \quad \frac{\partial}{\partial r_{ti}} (T_1 + T_2 + T_3) = 0,$$

for $r_{ti} = x_{ti}$ or $R = X$.

We must bear in mind, however, that R has been conditioned by $T_4 = \sigma^2$, which should be introduced in Lagrangean form into the expression to be minimized, i.e.

$$(16) \quad Q = T_1 + T_2 + T_3 + T_4 - \lambda (T_4 - \sigma^2)$$

so that $\partial Q / \partial r_{ti} = 0$ with $T_4 = \sigma^2$ are satisfied by $r_{ti} = x_{ti}$ and $\lambda = 1$.

So we have proved the intuitive result that the best linear estimator of the coefficient matrix β is the regression estimator b . No novelty is claimed: it is a Gauss-Mark-off property. The matrix treatment and the use of standardization of X may have some interest.

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