Most Efficient Least Squares Estimator in Multivariate Regression

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The model in matrix notation is

(1) $$\underset{(T1)}{y} = \underset{(Tk)}{X}\ \underset{(k1)}{\beta} + \underset{(T1)}{u}$$

with X standardized, i.e. $X'X = T1_k$. The regression coefficient estimator is

(2) $$b = (X'X)^{-1}X'y = \frac{1}{T}X'y = \beta + \frac{1}{T}X'u.$$

Let

(3) $$b_1 = \frac{1}{T}R'y$$

be any linear estimator of β. The object is to show that the minimum value of δ, where

(4) $$\delta = E(y_1 - \eta)'(y_1 - \eta)$$

with

(5) $$y_1 = Xb_1; \quad \eta = X\beta$$

is attained for R = X.

From (1), (3) and (5),

(6) $$y_1 = \frac{1}{T}XR'(X\beta + u).$$

Hence

(7) $$y_1 - \eta = -X\beta + \frac{1}{T}XR'X \quad + \frac{1}{T}XR'u,$$

so that, from (4),

(8) $$\varphi = -\beta'X'R\beta - \beta'R'X\beta + \frac{1}{T}\beta'X'RR'X\beta + \frac{1}{T}Eu'RR'u,$$

where φ is the value of the terms in R in δ(4).

We now propose finding the minimum value of φ from

(9) $$\frac{\partial \varphi}{\partial r_{ti}} = 0, \quad t = 1, 2, \ldots, T; \quad i = 1, 2, \ldots, k.$$

or, rather, showing that (9) is satisfied by $r_{ti} = x_{ti}$, these being the respective elements of R and X. Call the terms on the r.s. of (8) T_1, T_2, T_3, T_4. Clearly, w.l.g., T_4, which equals $\sigma^{-2}\mathrm{Tr}RR'$, can be taken as constant independent of R, e.g. $= \sigma^{2}$. Of course, $T_1 = T_2$. Set

(10) $$X\beta = Z = \{z_1, z_2, \ldots, z_T\}$$

it can then easily be shown that

(11) $$\frac{\partial T_1}{\partial r_{ti}} = -z_t\beta_i = \frac{\partial T_2}{\partial r_{ti}}$$

and not quite so easily that

(12) $$\frac{\partial T_3}{\partial r_{ti}} = \frac{2}{T} z_t \left(\sum_{s=1}^{T} z\, r_{si} \right)$$

with, from (10),

(13) $$z_s = \sum_{j=1}^{k} x_{sj}\beta_j.$$

On substitution for z_s, given by (13) in the brackets ( ) in (12), setting $r_{si} = x_{si}$ and using the orthogonal property of X, we find

(14) $$\frac{\partial T_3}{\partial r_{ti}} = 2z_t\beta_i.$$

Accordingly, from (11) and (14),

(15) $$\frac{\partial}{\partial r_{ti}}(T_1 + T_2 + T_3) = 0,$$

for $r_{ti} = x_{ti}$ or $R = X$.

We must bear in mind, however, that R has been conditioned by $T_4 = \sigma^2$, which should be introduced in Lagrangean form into the expression to be minimized, i.e.

$$(16) \qquad \varphi = T_1 + T_2 + T_3 + T_4 - \lambda(T_4 - \sigma^2)$$

so that $\partial\varphi/\partial r_{ti} = 0$ with $T_4 = \sigma^2$ are satisfied by $r_{ti} = x_{ti}$ and $\lambda = 1$.

So we have proved the intuitive result that the best linear estimator of the coefficient matrix β is the regression estimator b. No novelty is claimed: it is a Gauss-Mark-off property. The matrix treatment and the use of standardization of X may have some interest.

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