A Property of Conditioned Least Squares Estimation in Linear Multivariate Regression

In log form Cobb-Douglas is as follows

(1) $Y = a + b_1 X_1 + b_2 X_2 + v$

where Y, X_1 and X_2 are logs output, labour hours and capital stock respectively, v residual. Sometimes the coefficients b_1 and b_2 are estimated in unrestricted fashion, the fact of their sum being near unity being regarded as a desirable property. Less frequently they are constrained a priori by the law

(2) $b_1 + b_2 = 1,$

which is equivalent to stating that if labour and capital change in the same proportion so does output. This is the property of homogeneity.

If in (1) we substitute for b_2 the estimation of b_1 is transformed into a problem of simple regression

(3) $Y - X_2 = a + b_1 (X_1 - X_2) + v,$

when the LS estimate is

(4) $b_1 = \Sigma (y - x_2)(x_1 - x_2) / \Sigma (x_1 - x_2)^2$

where $y = Y - \overline{Y}$, $x_1 = X_1 - \overline{X}_1$, $x_2 = X_2 - \overline{X}_2$. But we could also have substituted for b_1 to get the LS estimate for b_2:-

(5) $b_2 = \Sigma (y - x_1)(x_2 - x_1) / \Sigma (x_2 - x_1)^2$

The property is: estimates (4) and (5) are consistent in the sense that their sum is exactly unity. This is evident from the formulae.

Full formal treatment of the problem would involve minimization of

(6) $2Z = \Sigma (a + b_1 X_1 + b_2 X_2 - Y)^2 - 2 \lambda (b_1 + b_2),$

where λ is a Lagrange. It is easy to show that formulae for b_1 and b_2 from this treatment are precisely those at (4) and (5).

A more general property would be: with k independent variables conditioned by m ($< k$) linear relations, let any m variables be expressed in terms of the remaining ($k - m$) and substituted in the regression equation, thus changing the problem into an unconditioned multivariate regression in ($k - m$) variables. The LS estimates of the coefficients of these variables are precisely those which would be found from full formal LS treatment using m Lagrange multipliers.

This property may not be quite trivial.

5 November 1969 R. C. Geary