A SIMPLE APPROACH TO MACRO-ECONOMIC DYNAMICS

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1. *Introduction*

Text-books on macro-economic theory generally comprise lengthy verbal expositions supported by simple (though often ingenious) diagrams and perhaps by an occasional footnote or appendix of mathematics. For this reason, if no other, the text goes little beyond the familiar Keynesian system of the short period and an introduction to the fashionable growth theory of the long period. This is all very well, but it is important to take the student quickly to what must be the essence of macro-economics: an analysis of dynamic systems on as wide and as general a basis as possible. It is here that mathematics becomes an essential element in the development and not just a convenient expository device. It is as well, therefore, to present even the simplest models of equilibrium and disequilibrium dynamics in mathematical terms from the outset. The mathematical formulation throws a powerful spotlight on the structure and important features of the models, showing up their precise limitations and guiding us in extending them in the direction of realism and econometric testing. The following sequence of highly simplified models is offered as the irreducible basis on which useful and realistic macro-dynamic theories can be constructed.

2. *The Variables and Relations*

The system of a closed economy is aggregated into one sector in real terms. In the circular flow of income in the system, it is assumed that there is no lag between output and income but that demand can lag behind income and output behind demand. There are then two real aggregates to specify: Y for income and output alike and Z for demand. Attention is first concentrated on the split of demand Z into the two constituents of consumption and investment, other elements being added later as the models are developed.

Two *ex ante* or planned functional relations are to be specified: one for consumption and one for investment. The simplest consumption function is taken, giving consumption C as dependent only on income Y, the effects of other factors (such as prices and incomes distribution) being ignored. A simple investment function is postulated, giving investment I as dependent only on the rate of change of output DY, ignoring such factors as profits, the stock of capital and the interest rate. In the continuous analysis adopted here, the rate of change over time is represented by the operator $D = d/dt$.

It is convenient to separate out the autonomous elements in consumption and investment and to write the functions generally as:

$$\text{Consumption} = b + C(Y); \qquad \text{Investment} = a + I(DY),$$

where a is autonomous investment, not dependent on changing output, and b is autonomous consumption, broadly corresponding to the subsistence level. Then $C(Y)$ vanishes when Y is zero and $I(DY)$ when Y is constant. Total autonomous expenditure is written $A = a + b$. Hence, apart from any other constituents of demand which may be added later, and apart from any lags which we may wish to assume, we have aggregate demand:

$$Z = C(Y) + I(DY) + A$$

comprising planned consumption and investment together with autonomous expenditure of all kinds. In general, the functions C and I need to be specified, e.g. in terms of appropriate parameters.

The special case presented here, in developing linear models of macrodynamics, is that in which both consumption and investment functions are linear. Apart from lags:

$$(1) \qquad Z = cY + vDY + A, \qquad \text{where} \quad s = 1 - c \ (0 < s < 1).$$

There are two parameters, the constant marginal propensity to save s and the constant ratio v of desired investment to change in output. Note that v is the incremental form of the (desired) capital/output ratio. It will be referred to as such for convenience though no explicit reference is made to the capital stock. The only limitation imposed is that the ratio is positive ($v > 0$).

3. *The Basic Macro-dynamic Model*

All the models considered are dynamic since investment depends on the change in output over time. The solution of any model is obtained when we have the path of output Y over time. The models are also dynamic in the sense that, generally, we assume lags in the planning processes. With lags in both constituents of demand and in the determination of output, our basic model is characterized by two conditions, one a lagged version of demand (1) and the other the lagged relation of output to demand:

$$(2) \qquad \begin{array}{ll} \text{Demand:} & Z = C + I + A \\ \text{Output:} & Y \text{ lagged on } Z, \end{array}$$

where C is lagged on cY and I is lagged on vDY.

The form taken for a lag is important; it has a critical influence on the solution of a model.[1] In the basic models developed here only the simplest continuous form of lag is taken, that known as the single exponential lag. For output lagging on demand, for example, the assumption is that output adjusts continuously to the deficit of current output below demand:

$$DY = \lambda(Z - Y), \quad \text{where } \lambda > 0 \text{ is the speed of response.}$$

We take the parameter in reciprocal form, $L_0 = 1/\lambda$, for the time-constant or *length of lag*. Hence the exponential lag of Y on Z is obtained in operator form:

$$Y = \frac{1}{L_0 D + 1} Z \quad \text{for a given length of lag } (L_0 > 0).$$

Similarly, write $C = \dfrac{1}{L_c D + 1} cY$ and $I = \dfrac{1}{L_i D + 1} vDY$.

Of the planning processes, that for investment is the one most obviously lagged. Our simple models generally include a lag in investment if no other. Attention is then to be concentrated on the length of this lag, i.e. on the influence of shorter or longer lags on the path of output. As our main parameter, therefore, we write L (instead of L_i as above) for the length of the investment lag. The other parameters are s, v and any other lag lengths introduced into the model.

The models of basic form (2) are appropriately classified into two categories:

If the only lag is in investment, then the model (2) becomes:

(3) $Z = (1-s)Y + I + A$ and $Y = Z$,
i.e. $sY = I + A$, where I is lagged on vDY.

We have then a model subject to the simple condition that planned saving and investment are always equal over time. All equilibrium models are of this type and also those disequilibrium models in which there is a lag in investment but no other.

If there are other lags, then the form of the model remains as shown in (2) and there is no planned saving/investment equality. Here we must have a disequilibrium model, one in which there can be unintended saving and/or investment. If there is a consumption lag with output increasing, then there is unintended saving because of the planning of consumption on the basis of earlier (and lower) income levels. Similarly, if there is an output lag in an expanding economy, there is unintended disinvestment (in stocks) because output is planned on the basis of earlier (and lower) demand.

[1] See A. W. Phillips, "Stabilisation Policy and the Time-Form of Lagged Responses", *Economic Journal*, vol. LXVII (1957).

As we shall see, the distinction between the two categories of models is a very fruitful one. The equilibrium-type models of the first category can be viewed in the long period as the basis of various types of economic growth. The disequilibrium models of the second category are particularly relevant to the shorter period and to the analysis of oscillations.

4. *Growth Models*

Take a single exponential lag of length L in investment and no other lag in the system. By (3) the model is:

$$sY = \frac{1}{LD+1} vDY + A, \qquad \text{i.e.} \quad (v - sL)DY - sY + A = 0.$$

Autonomous expenditure A is generally given as some specified function of time (e.g. a growth at rate ρ). In the case A constant over time, the path of Y (as the solution of the differential equation) is found to be:

$$(4) \qquad Y = A/s + (Y_0 - A/s)e^{gt}, \qquad \text{where} \quad g = s/(v - sL)$$

for a specified initial value Y_0.

The result (4) needs careful interpretation. First, $Y = A/s$ (constant over time) is a solution; it is the path of output generated if the initial output happens to be $Y_0 = A/s$. This is the *stationary level* of output, that given on multiplying up autonomous expenditure A by the Keynesian multiplier $(1/s)$. There is no other investment and output is constant at the stationary level over time.

Second, the general solution (4) for $Y_0 \neq A/s$ is some variation about the stationary level. The sign of g determines whether there is a progressive divergence from the stationary level $(g > 0)$ or a steady convergence back towards the stationary level $(g < 0)$. These are the only two possibilities. Given the parameters s and v, the length L of the investment lag is decisive:

Divergence for short investment lags: $L < v/s$
Convergence for long investment lags: $L > v/s$.

The case $L = v/s$ exactly, or approximately, is that of resonance. We can note the possibility, but it is of little significance since it depends on accidental agreement between the parameters of the system.

Third, for long investment lags $(L > v/s)$ we have a model which simply establishes the stability of the stationary level, stability being in the sense of disequilibrium (lagged) dynamics. With $g < 0$, the path (4) when $Y_0 > A/s$ gives a decline in output back towards the level A/s and when $Y_0 < A/s$ a rise in output back towards this level. In either case $Y \to A/s$ as time goes on.

Fourth, for short investment lags $(L < v/s)$, we have progressive growth in output at the constant proportional rate $g = s/(v - sL)$ away from the level A/s, provided that $Y_0 > A/s$. This is the well-known simple model of *steady-state growth* at the "warranted" rate g. In the special case when $L=0$, and the investment lag is so short that it can be ignored, then we have the *Harrod-Domar growth model* and the warranted rate of growth $g=s/v$. The Harrod-Domar model is one of equilibrium; there are no lags at all in the system. What we have established is that a similar result is obtained in a disequilibrium model provided that the only lag in the system is one of simple exponential form in investment. The growth rate is then greater than the Harrod-Domar warranted rate ($g=s/v$); it depends on, and increases with, the length of the investment lag. From this point of view, such disequilibrium models are very much akin to models of equilibrium.

Fifth, if we look at the model from the opposite angle, we can start with the Harrod-Domar model, and the warranted rate ($g=s/v$) is the equilibrium rate of growth in the absence of all lags. We can then obtain a related disequilibrium model by introducing a simple lag in investment of length L. The result is that, as L increases from zero to a critical value (v/s), the rate of growth increases away from the equilibrium rate. On this disequilibrium model, we have established the instability of the Harrod-Domar equilibrium path.

Sixth, when the investment lag is short ($L<v/s$ and $g>0$), suppose that initially output is below the stationary level: $Y_0<A/s$. Then the path of output (4) is a progressive *decline* of output away from A/s at the warranted rate g. This implies that Y soon sinks to zero (and below) and that there is progressive disinvestment. This is a completely unrealistic case which we can simply ignore by assuming that output is above the stationary level and by concentrating on the situation of progressive investment and growth in output.

Finally, we can ask: what is the likely length of the investment lag? Take the year as the time-unit. Even if the capital/output ratio v is as low as 1 on annual date and s is as high as 0·25, the case of convergence only arises if $L>4$ years. It would seem that, for reasonable values of s and v and for fairly short investment lags, we have here a model of growth. The fact that the model is so simple as to be unrealistic needs no emphasis. The difficulties are well known: the fact that the growth rate is unstable in almost any disequilibrium version (the "knife-edge" problem) and the need for "accidental" agreement between the warranted growth rate g in output and the rate of growth in resources which can sustain it (e.g. the growth in the labour force). Nevertheless, the simple model provides the basis on which a fine crop of more sophisticated models has been produced.

5. *Disequilibrium Models of Oscillations*

As the next simplest model, take a single exponential lag of length L in investment and one other lag, which we opt to take as a single exponential lag of length TL in output. The second lag parameter T is the ratio of the length of the output lag to that in investment and, in view of the relative importance of the investment lag, we may assume $0<T<1$. We now have a disequilibrium model and by (2) it is:

$$Z=(1-s)Y+\frac{1}{LD+1}vDY+A \quad \text{and} \quad Y=\frac{1}{TLD+1}Z \tag{5}$$
$$\text{i.e.} \quad Y=\frac{1}{TLD+1}\left[(1-s)Y+\frac{1}{LD+1}vDY+A\right].$$

Taking all lags through to output, we can interpret the model as having a single lag in consumption and a double lag in investment, keeping the relative dominance of the lag in the investment process. The main parameter is still L; there are three other parameters: s, v and T.

Generally A varies over time but, in the case A constant, the differential equation (5) becomes:

$$TL^2D^2Y-[v-(T+s)L]DY+sY=A. \tag{6}$$

The solution for the path of Y over time is now more varied. It can show oscillations as an alternative to (but not as well as) steady growth in output, according to the values of the parameters. The solution of (6) gives:

Range of values of L	*Path of output*
$L<\frac{v}{(\sqrt{T}+\sqrt{s})^2}$	Steady-state growth from $Y=A/s$
$\frac{v}{(\sqrt{T}+\sqrt{s})^2}<L<\frac{v}{T+s}$	Explosive oscillations about $Y=A/s$
$\frac{v}{T+s}<L<\frac{v}{(\sqrt{T}-\sqrt{s})^2}$	Damped oscillations about $Y=A/s$
$L>\frac{v}{(\sqrt{T}-\sqrt{s})^2}$	Steady convergence to $Y=A/s$

For investment lags of increasing length, given s, v and T, the path of output changes from steady-state growth (when the investment lag is short, as in the previous model) to explosive oscillations about the stationary level, then to damped oscillations and finally to convergence to the stationary level.

The question is: what is the situation likely to be in practice? Is an oscillatory path the probable one? The broad answer is that it is. As illustration, suppose as before that v is as low as 1 on annual date and s as high as 0·25. Further, take $T=9/16$ for an output lag of a little over half the investment lag in length. Then the three critical values of L in the above table are respectively about 8 months, about 15 months, and 16 years. A very short investment lag, as in the Harrod-Domar model, gives a steady-state growth path. But, over the wide range from 8 months to 16 years, investment lags correspond to oscillatory paths of output in this model. A reasonably short lag of a little more than a year is to be expected and the oscillation then is mildly explosive or damped. The model still has unrealistic features; for example, the nearly regular oscillation in our illustration has a long period, of some 10 years in length.

We have, essentially, a model of oscillations. Whereas before we looked for an investment lag short enough to be consistent with steady-state growth in long-period equilibrium, we now depend on reasonably long investment lags, combined with shorter output lags, to throw up oscillations in output in shorter-period disequilibrium. As the oscillations work themselves out, output falls below capacity and then recovers: there is unintended disinvestment and then unintended investment.[1]

6. *Stabilization Policies in a Closed Economy*

The construction of our models is an exercise in positive economics, an attempt to describe how the economic system works. It is another matter to try to control the working of the system and to improve on the outcome. It is time to go on from positive economics to considerations of optimalization and so to economic policy.

Even in our simple models of growth and oscillations in a closed economy, the problem of control is a complicated one. Suppose that our model gives a path of output which is an oscillation about a stationary level. There is a double problem: the stationary level may be too low (for full employment) and needs to be raised; the oscillations may be undesirable and need to be eliminated. Economic policy has a corresponding double task, to optimalize and to stabilize the level of output.

The stationary level of output in our models is $Y=A/s$. Let Y^* be the full-employment level of output, fixed exogenously by the size of the labour force. Then $A^*=sY^*$ is the appropriate autonomous expenditure to give Y^* as the stationary level of output. But except by accident —or control—the actual autonomous expenditure $A \neq A^*$ and the model is not consistent with full employment.

Since there is a single aggregate A on which to act, the policy-maker does not seem to have a difficult task, at least as regards level as opposed to oscillations in output. He has only to add (or subtract) the appropriate amount of government demand, to increase purchasing power (or remove some of it) by direct or indirect official action. But he is not likely to hit the nail exactly on the head every time. So, if too much

[1] The model has oscillations as an alternative to growth. To combine the two requires further elaboration. A "forced" growth can be added to the oscillatory path by taking autonomous expenditure, A in the model, as growing at some given exogenous rate. But this provides no explanation of growth. To do so, in a fully articulated model of cyclical growth, would seem to require a model wide enough to comprise the money market as well as real factors and to have only the money supply as exogenous. See R. G. D. Allen, *Macro-Economic Theory*, 1967, ch. 20, and the work of Phillips and Bergstrom there cited.

purchasing power is pumped in, output tends to rise above the full-employment level and the reverse process of removing purchasing power is needed. This may again be overdone and output tends to fall below the full-employment level, requiring another reversal of policy. There are here some of the all-too-familiar features of stop/go.

Something more sophisticated, and more continuously automatic, is needed. The engineer has installed control systems in industrial processes and plants. His methods need to be adapted and applied to the flows of the economic system as they are to those in a refinery or distillery. It is easy enough to suggest what needs to be controlled. If aggregate output Y is below the desired level Y^*, controls should operate to raise Y; if the rate of increase of Y is too low, they should act to increase the rate. It is also fairly easy to work out the operation of the controls in simple macro-dynamic models.[1] Take the model of oscillations described by the differential equation (5) as an illustration. The control needed is to add an element of official demand to that given in (5), making official demand proportional always to the deficit of output $(Y^* - Y)$, either at the current time or cumulated over time or both. Since it takes time to decide to pump in (or to remove) purchasing power and still more time for the necessary action to be taken, the automatic official demand is to be added to (5) with an appropriate lag. On solving the resulting differential equation, we find that the stationary level of the system can indeed be adjusted to the desired level Y^*, achieving the optimal objective.

All this ignores the oscillatory nature of the system based on the model (5). And, indeed, it is found that the control suggested tends to increase rather than to diminish the oscillations in the system. The other objective, that of stabilization, remains to be achieved. A second form of automatic official demand can be applied for this purpose, i.e. demand proportional to $D(Y^* - Y)$, the rate of change of the deficit. We find this control can at least damp down, if not eliminate completely, the oscillations in output around Y^*. The double trick of optimalization and stabilization can be nearly turned, in theory and for this particular model. It remains to establish the possibility of control in practice and with a realistic model of the economy. That this is far from easy is seen in the stress which Phillips lays on the fact that the results of control are very sensitive to changes in the parameters of official demand.

7. *Open-economy Models*

The models we have described relate to a closed economy and we have seen how stop/go policies can result. But stop/go in its most persistent and aggravated forms is a feature of an open economy in which the balance of external payments conflicts with the domestic objective of full employment. It remains to extend the models to make them apply

[1] See A. W. Phillips, "Stabilisation Policy in a Closed Economy", *Economic Journal*, vol. LXIV (1954), on which the following analysis is based.

to an open economy with external trade. The following is a simple extension on lines suggested by Stone.[1]

In an open economy, there are exports X and imports M so that demand is:

$$Z = C + I + A + X - M.$$

In the absence of lags, take $C=(1-s)Y$ and $I=vDY$ in their usual linear forms. On the external side, we assume for our extended model that X is given exogenously and that M is proportional to output: $M=\mu Y$, where μ is a constant marginal propensity to import. Then without lags:

$$Z=(1-s)Y+vDY+A+X-\mu Y$$
$$\text{i.e.}\quad Z=[1-(s+\mu)]Y+vDY+(A+X).$$

Hence the only difference is that the marginal propensity to save, s, is replaced by the larger parameter $(s+\mu)$, the combined propensity to save and to import, and the autonomous expenditure A is increased to $(A+X)$ by the addition of exports. The same is true of any model, including those of Sections 4 and 5 above, in which consumption and imports are subject to no (or the same) lag.

The growth model, with (4) as the path of output, extends at once to the open economy assumed here. The stationary level of output becomes $(A+X)/(s+\mu)$ and the warranted growth rate becomes:

$$g=\frac{s+\mu}{v-(s+\mu)L}>\frac{s}{v-sL}.$$

The stationary level $(A+X)/(s+\mu)=A/s$ if $X=\mu(A/s)$ and so equal to imports. If X is not far from $\mu A/s$, and the external deficit or surplus small, the stationary level is not very different from that of the closed economy. However, the opening of the economy seems always to increase the steady-state growth rate. In the special case $(L=0)$ of the Harrod-Domar model, the growth rate is increased from s/v to $(s+\mu)/v$.

Equally, the oscillatory model, with (6) as the differential equation, extends at once. The differential equation becomes:

$$TL^2D^2Y-[v-(T+s+\mu)]DY+(s+\mu)Y=A+X$$

when both A and X are constant over time. The three critical values of L, for oscillations of explosive and damped forms, become:

$$\frac{v}{[\sqrt{T}+\sqrt{(s+\mu)}]^2};\quad \frac{v}{T+(s+\mu)};\quad \frac{v}{[\sqrt{T}-\sqrt{(s+\mu)}]^2}.$$

The last remains large for all reasonable values of the parameters. The main effect of the addition of μ in the open-economy model is to reduce the values of the first two critical L's. Oscillations are more likely, even

[1] Richard Stone, "Our Unstable Economy: Can Planning Succeed?". Sixth Annual Lecture of the U.K. Automation Council, 1966.

for small investment lags, and damping is more usual. In our illustrative case, with $v=1$, $s=0{\cdot}25$ and $T=9/16$, we found that oscillations occur in the closed economy for investment lags greater than about 8 months and damping for lags of about 15 months or more. Suppose that, in the open-economy model, $\mu=7/36$. The two critical lengths of investment lag are then approximately 6 months and 12 months respectively.

Stabilization policies can now be approached on the lines already laid down, but with one extra and very important feature. We start by assuming that X is fixed and that only domestic autonomous expenditure A is subject to official policy manipulation. If Y^* is the full-employment level of output then the stationary level of output in the open-economy models corresponds if $(A+X)/(s+\mu)=Y^*$. Hence, the required domestic expenditure $A^*=(s+\mu)Y^*-X$, which is not very different from the previous amount (sY^*) *provided* that X is near to μY^*, the value at which exports balance imports at full-employment output. Domestic economic policy aims at achieving A^*. The new feature is that the surplus or deficit on external account also needs to be examined.

At the stationary level of output, the external surplus is:

$$X-\mu Y=X-\mu(A+X)/(s+\mu)=(sX-\mu A)/(s+\mu).$$

If $A>sX/\mu$, then there is an external deficit and reserves decline. If $A<sX/\mu$, the opposite is the case and reserves increase. A constant level of reserves obtains if and only if $A=sX/\mu$.

It is impossible to fix A so that *both* full-employment output is achieved *and* the external position is in balance. Suppose that $X<\mu Y^*$ and given exogenously. Then, for full employment, A must be set at the level $(s+\mu)Y^*-X>sY^*>sX/\mu$ and reserves decline with corresponding balance-of-payments difficulties. On the other hand, for constant reserves, A must be set at the level $sX/\mu<sY^*$, and this is too low for full employment at home. The stop/go features of official policy directed at domestic expenditure A are even more prominent in the situation of the open-economy model. The nearer A is set to achieve full employment, the more out of balance does the external account become, and conversely emphasis on the balance of payments results in less-than-full employment at home.

On the other hand, if both domestic expenditure A and exports X are amenable to change by government policy, then the double objective of full employment and a balance in the external account can be achieved in the stationary level of output. It is only necessary to set $A=sY^*$ and $X=\mu Y^*$. Then the stationary level of output is $(A+X)/(s+\mu)=Y^*$ for full employment, and the external surplus at this output is $X-\mu Y^*=0$, so that the external account is balanced and reserves are constant.

The problem of stabilization is then much as before: how to set the appropriate A and X, and at the same time to damp down oscillations without recourse to stop/go. The broad conclusion is that an automatic stabilization control system is needed, on lines already outlined but extended to two control variables: $(X-\mu Y)$ as well as (Y^*-Y).

8. *Saving Model: Optimal Allocation over Time*

We turn finally to a different kind of model, that of optimalization over time with particular reference to the allocation of income between consumption and saving. The simplest form of the model was suggested nearly 40 years ago by a mathematician, F. P. Ramsey, and largely ignored until revived quite recently as a basis for current investigations into macro-economic optimalization.[1]

The criterion selected for the optimal process is the aggregate net utility obtained by consumers over time, i.e. the utility of consumption less the disutility of labour, and for an optimal position to exist we must assume that there is an upper bound to the net utility obtainable from economic activity in the system. Further, the optimalization process is sufficiently complicated in itself, so that to obtain a simple model we assume that there are no lags in the economy. The basic macro-dynamic model, to which the new model relates, is then that of Harrod-Domar with growth in output at the warranted rate $g=s/v$, where $s=S/Y$ is a constant fraction of income saved and v is a constant capital/output ratio.

In view of the fact that the present model sets the disutility of labour against the utility of consumption, we make first a change in the formulation of the basic model without modification of the solution. The investment function is dropped in favour of a production function taken in fixed-coefficients form: $Y=K/v=L/u$, where a homogeneous capital stock K is now assumed, used in conjunction with labour input L. The constant v is a capital/output ratio, now taken in the form of capital-stock/capacity-output. Then $K=vY$ and investment $DK=vDY$ as before, as long as full-capacity operation is maintained. Adding a linear consumption function and assuming no lags, we obtain the Harrod-Domar growth model. The other constant u then serves to give the demand for labour $L=uY$ from the output Y. The next step is to extend the formulation to a general production function $Y=F(K, L)$, where F is unspecified but subject to $\partial Y/\partial K=F_K>0$ and $\partial Y/\partial L=F_L>0$. This limitation is that the marginal products are positive.

The essential change in the basic model is the introduction of the optimalization process. The consumption function, which implies a separate consumption/saving allocation at each point of time, is dropped in favour of a planned allocation over time which maximizes net utility in the aggregate from the present $(t=0)$ to $t=T$, where T is the time

[1] The original article is F. P. Ramsey, "A Mathematical Theory of Saving", *Economic Journal*, vol. XXXVIII (1928); and a simplified account, in the context of the calculus of variations, is given in R. G. D. Allen, *Mathematical Analysis for Economists*, 1938, pp. 537–40. On the recent revival, see a 1960 article by Richard Stone on "Three Models of Economic Growth", reproduced in his *Mathematics in the Social Sciences and Other Essays*, 1966, and J. R. Hicks, *Capital and Growth*, 1965, ch. XXI.

horizon of consumers. We assume that net utility is separately and independently determined at each point of time and then aggregated over time.[1] Write $U=U(C, L)$ for the net utility of consumption C and labour L at time t so that the criterion is:

$$\int_0^T U(C, L)\, dt = \max.$$

To simplify we assume that T is infinite (an infinite time horizon) and that the utility of consumption $\phi(C)$ and the disutility of labour $\psi(L)$ are separable and additive in U:

$$U=\phi(C)-\psi(L)$$

where the marginal utility and disutility are positive: $\phi'(C)>0$, $\psi'(L)>0$.

All that remains is to assemble the conditions of the model by adding the usual demand and output conditions in the absence of lags and with no autonomous expenditure:

$$Z=C+DK \qquad \text{and} \qquad Y=Z.$$

The model is specified:

$$\int_0^\infty U\, dt=\max \qquad \text{when} \quad U=\phi(C)-\psi(L)$$

$$\text{subject to} \quad Y=C+DK \quad \text{and} \quad Y=F(K, L)$$

So:

$$(7) \qquad \int_0^\infty [\phi(Y-DK)-\psi(L)]\, dt=\max \qquad \text{subject to} \quad Y=F(K, L).$$

This is a problem in the calculus of variations. The time paths of capital and labour inputs (K and L) are to be determined, given initial conditions at $t=0$ and the maximum U^* of net utility obtainable in the system of any time, to achieve the optimum of (7). The time paths of output $Y=F(K, L)$, of consumption $(Y-DK)$ and of saving (and investment) DK then follow.

The optimal conditions (Euler's equations) give:

$$(8) \qquad \partial Y/\partial K=-D\phi'(C)/\phi'(C) \qquad \text{and} \qquad \partial Y/\partial L=\psi'(L)/\phi'(C),$$

and further the common value of saving/investment at any time in the optimal solution:

$$(9) \qquad S=DK=(U^*-U)/\phi'(C).$$

These results are surprisingly simple and useful; as Stone observes: "We seem to have got a lot out without putting very much in".[2] The results (8) state that, at each point of time, production is pushed to the

[1] This independence assumption is the most restrictive in simple optimum models of the Ramsey type. See Hicks, *op. cit.*, p. 261.

[2] Stone, *op. cit.*, p. 12.

point where the marginal product of capital (or the profit rate ρ under perfect competition) equals the proportional rate of decline in the marginal utility of consumption, and where the marginal product of labour (or the wage rate w under perfect competition) equals the ratio of the marginal disutility of labour to the marginal utility of consumption. The result (9) states that the split between consumption (C) and saving (S) is continuously adjusted over time so that:

$$\text{Saving} \times \text{marginal utility of consumption} = \text{deficit of } U \text{ below } U^*.$$

The greater the deficit of utility, the more consumption is postponed in favour of saving.

It is indeed surprising how far we have got with general functions. The results can even be generalized a little. For example, the optimal criterion assumed takes aggregate net utility over time without discounting. If instead there is discounting at the rate ρ, so that $\int_0^\infty Ue^{-\rho t}\,dt$ is maximized, the only modification is that ρ is added to the expression for $\partial Y/\partial K$ in (8). But, even so, the actual and limiting time paths of the variables can only be got by taking particular forms for the three general functions, F, ϕ and ψ. As Stone shows,[1] some striking results are obtained, particularly in the limit as time goes on, if we take the forms:

$$Y=\rho K+wL; \qquad \phi(C)=A-a/C^{\alpha}; \qquad \psi(L)=BL^{\beta}.$$

All the parameters in these forms are taken as positive and $\beta>1$. The time paths of income (output), of consumption and of saving then all grow, in the limit, at the steady-state rate $g=\rho/(1+\alpha)$. The stock of capital also has the same limiting rate of growth. A particular but incidental feature is obtained from the fact that the production function assumed is such that all output may be obtained from the capital stock and the input of labour may decline to zero. The working out of the optimal allocation of inputs over time is such that, as capital is built up (by saving and investment), the amount of labour used declines correspondingly at the rate $\rho/(\beta-1)$ and tends to zero.

With these particular forms of the functions, we can provide, finally, a link with the basic Harrod-Domar model. The saving/income and capital/output ratios vary over time on the optimal path. But, in the limit as time goes on, each tends to a constant value:

$$S/Y=1/(1+\alpha) \qquad \text{and} \qquad K/Y=1/\rho.$$

Hence, in the limit, the steady-state rate of growth in the system $g=\rho/(1+\alpha)=s/v$, where s is the constant fraction of income saved and v is the capital/output ratio. The optimalization process works itself out in the end to the warranted rate of growth of the Harrod-Domar model.

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[1] *Op. cit.*, pp. 61–4.